

# A Remark on Zeros of Brownian Motion

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July 9, 2009

## Abstract

Let  $\{W(t), t \geq 0\}$  be a standard Brownian motion. If  $I$  is a bounded interval on which  $W$  has no zero, an almost sure lower bound to  $\inf\{|W(t)|, t \in I\}$  can be provided, when  $I$  is taken from a given countable family of intervals covering the positive half-line.

## 1 Main Result

Let  $\{W(t), t \geq 0\}$  be a standard Brownian motion. Let  $I$  be some bounded interval of  $\mathbf{R}^+$ . Suppose  $W(t) \neq 0$ , for all  $t \in I$ . What can be said about the size of  $\inf\{|W(t)|, t \in I\}$ ? This one only depends on the location of  $I$  and of the size of  $I$ . The object of this note is to prove more precisely the following result.

**Theorem 1** *Let  $\vartheta_k \geq 0$  be such that  $T_N = \sum_{k \leq N} \vartheta_k \uparrow \infty$  and denote  $I_N = [T_N, T_{N+1}]$ . Let  $\eta_k \geq 0$  be such that*

$$\sum_{N \geq 1} \eta_N \min \left( \frac{1}{\sqrt{T_{N+1} - T_N}}, \frac{1}{\sqrt{T_N}} \right) < \infty$$

*Then*

$$\mathbf{P} \left\{ \inf_{t \in I_N} |W(t)| \geq \eta_N \text{ or } W(t) = 0 \text{ for some } t \in I_N, \quad N \text{ ultimately} \right\} = 1.$$

The proof relies upon several intermediate results on infima of  $|W|$ , which are also of independent interest.

## 2 Local infima of Brownian motion

In this section, we collect some properties of the infimum of  $W$  over bounded intervals. Precise estimates of the probability

$$\mathbf{P} \left\{ \inf_{a \leq t \leq b} |W(t) - M| \geq c \right\}.$$

will be necessary. Notice preliminary, since  $-W$  and  $W$  have same law that

$$\begin{aligned} \mathbf{P} \left\{ \inf_{a \leq t \leq b} |W(t) + M| \geq c \right\} &= \mathbf{P} \left\{ \inf_{a \leq t \leq b} | -W(t) + M| \geq c \right\} \\ &= \mathbf{P} \left\{ \inf_{a \leq t \leq b} |W(t) - M| \geq c \right\}, \end{aligned}$$

so that it is enough to consider the case  $M \geq 0$ . Put

$$\Psi(x) = \mathbf{P}\{W(1) > x\} = \int_x^\infty e^{-u^2/2} \frac{du}{\sqrt{2\pi}}, \quad x \in \mathbf{R}.$$

The lemma below is certainly well-known, although we could not find a reference. We included a proof for the sake of completeness.

**Lemma 2** *Let  $0 < a < b < \infty$ . Then for any  $c > 0$  and any real  $M$*

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c\right\} = \int_{|v| > c} \left[1 - 2\Psi\left(\frac{|v| - c}{\sqrt{b-a}}\right)\right] \frac{e^{-\frac{(M+v)^2}{2a}}}{\sqrt{2\pi a}} dv.$$

*Proof.* By symmetry of the law of  $W$  it suffices to consider the case  $M \geq 0$ . By the intermediate values Theorem,

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c\right\} = \mathbf{P}\left\{\inf_{a \leq t \leq b} W(t) \geq M+c\right\} + \mathbf{P}\left\{\sup_{a \leq t \leq b} W(t) \leq M-c\right\}. \quad (1)$$

Let  $x \geq 0$ . Then  $\mathbf{P}\{\inf_{a \leq t \leq b} |W(t) - M| \geq c \mid W(a) = M \pm x\} = 0$ , if  $0 \leq x \leq c$ ; and if  $x > c$ ,

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c \mid W(a) = M+x\right\} = \mathbf{P}\left\{\sup_{a \leq t \leq b} (W(a) - W(t)) \leq x-c\right\},$$

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c \mid W(a) = M-x\right\} = \mathbf{P}\left\{\sup_{a \leq t \leq b} (W(t) - W(a)) \leq x-c\right\}.$$

As for  $y \geq 0$ , ([1], Theorem 1.5.1)

$$\mathbf{P}\left\{\sup_{0 \leq t \leq T} W(t) > y\right\} = 2\mathbf{P}\{W(T) > y\},$$

we get if  $|x| > c$ ,

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c \mid W(a) = M+x\right\} = 1 - 2\Psi\left(\frac{|x| - c}{\sqrt{b-a}}\right). \quad (2)$$

Therefore

$$\begin{aligned} & \mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c\right\} \\ &= \int_{\mathbf{R}} \mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c \mid W(a) = u\right\} \frac{e^{-\frac{u^2}{2a}}}{\sqrt{2\pi a}} du \\ &= \int_{|u-M| > c} \mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c \mid W(a) = u\right\} \frac{e^{-\frac{u^2}{2a}}}{\sqrt{2\pi a}} du \\ &= \int_{|u-M| > c} \left[1 - 2\Psi\left(\frac{|u-M| - c}{\sqrt{b-a}}\right)\right] \frac{e^{-\frac{u^2}{2a}}}{\sqrt{2\pi a}} du \\ &= \int_{|v| > c} \left[1 - 2\Psi\left(\frac{|v| - c}{\sqrt{b-a}}\right)\right] \frac{e^{-\frac{(M+v)^2}{2a}}}{\sqrt{2\pi a}} dv, \end{aligned} \quad (3)$$

as claimed. ■

**Remark 3** It follows from Lemma 2 that

$$\begin{aligned}\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| > 0\right\} &= \lim_{c \downarrow 0} \mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq c\right\} \\ &= \int_{\mathbf{R}} \left[1 - 2\Psi\left(\frac{|v|}{\sqrt{b-a}}\right)\right] \frac{e^{-\frac{(M+v)^2}{2a}}}{\sqrt{2\pi a}} dv. \quad (4)\end{aligned}$$

Thus

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| \geq 0\right\} = 1 \neq \int_{\mathbf{R}} \left[1 - 2\Psi\left(\frac{|v|}{\sqrt{b-a}}\right)\right] \frac{e^{-\frac{(M+v)^2}{2a}}}{\sqrt{2\pi a}} dv,$$

yielding a discontinuity at 0. Take for instance  $a = 1$ ,  $b = 1 + \mu^2$ ; the integral above is

$$\int_{\mathbf{R}} \left[1 - 2\Psi\left(\frac{|v|}{\mu}\right)\right] \frac{e^{-\frac{(M+v)^2}{2}}}{\sqrt{2\pi}} dv \rightarrow 0 \quad \mu \rightarrow \infty.$$

And

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| = 0\right\} = 2 \int_{\mathbf{R}} \Psi\left(\frac{|v|}{\sqrt{b-a}}\right) \frac{e^{-\frac{(M+v)^2}{2a}}}{\sqrt{2\pi a}} dv. \quad (5)$$

We will also show

**Lemma 4** *There exists an absolute constant  $C$ , such that for every real  $M$ , and  $0 < a < b < \infty$ ,*

$$\begin{aligned}\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| = 0\right\} &= 2 \int_{\mathbf{R}} \Psi\left(\frac{|v|}{\sqrt{b-a}}\right) \frac{e^{-\frac{(M+v)^2}{2a}}}{\sqrt{2\pi a}} dv \\ &\leq C \min\left(1, \sqrt{\frac{b-a}{a}} e^{-\frac{M^2}{8 \max(a, b-a)}}\right).\end{aligned}$$

*In particular*

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| = 0\right\} \leq C \min\left(1, \sqrt{\frac{b-a}{a}}\right).$$

*Proof.* By (2.6)

$$\begin{aligned}&\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| = 0\right\} \\ &= 2 \int_{\mathbf{R}} \Psi\left(\frac{|v|}{\sqrt{b-a}}\right) \frac{e^{-\frac{(M+v)^2}{2a}}}{\sqrt{2\pi a}} dv \\ &= 2 \left\{ \int_{|v| \leq \frac{M}{2}} + \int_{|v| > \frac{M}{2}} \right\} \Psi\left(\frac{|v|}{\sqrt{b-a}}\right) \frac{e^{-\frac{(M+v)^2}{2a}}}{\sqrt{2\pi a}} dv \\ &\leq \sqrt{\frac{2}{\pi a}} \left\{ e^{-\frac{M^2}{8a}} \int_{\mathbf{R}} \Psi\left(\frac{|v|}{\sqrt{b-a}}\right) dv + \int_{|v| > \frac{M}{2}} \Psi\left(\frac{|v|}{\sqrt{b-a}}\right) dv \right\} \\ &= \sqrt{\frac{2}{\pi a}} \left\{ e^{-\frac{M^2}{8a}} \sqrt{b-a} \int_{\mathbf{R}} \Psi(|w|) dw + \int_{|v| > \frac{M}{2}} \Psi\left(\frac{|v|}{\sqrt{b-a}}\right) dv \right\}. \quad (6)\end{aligned}$$

Recall that the Mills' ratio  $R(x) = e^{x^2/2} \int_x^\infty e^{-t^2/2} dt = (\sqrt{2\pi})e^{x^2/2}\Psi(x)$  verifies for all  $x \geq 0$   $R(x) \leq \sqrt{\pi/2}$ . Thus

$$\begin{aligned} \int_{|v| > \frac{M}{2}} \Psi\left(\frac{|v|}{\sqrt{b-a}}\right) dv &= \sqrt{b-a} \int_{|w| > \frac{M}{2\sqrt{b-a}}} \Psi(|w|) dw \\ &\leq C\sqrt{b-a} \int_{|w| > \frac{M}{2\sqrt{b-a}}} e^{-w^2/2} dw \\ &\leq C\sqrt{b-a} e^{-\frac{M^2}{8(b-a)}}. \end{aligned} \quad (7)$$

Therefore

$$\begin{aligned} \mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| = 0\right\} &\leq C\sqrt{\frac{b-a}{a}} \left\{e^{-\frac{M^2}{8a}} + e^{-\frac{M^2}{8(b-a)}}\right\} \\ &\leq C\sqrt{\frac{b-a}{a}} e^{-\frac{M^2}{8\max(a, b-a)}}. \end{aligned} \quad (8)$$

■

One can recover as a special case that (see [2] p.248)

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t)| = 0\right\} = 1 - \frac{2}{\pi} \arctan \sqrt{\frac{a}{b-a}},$$

or, equivalently  $\mathbf{P}\{W(t) \text{ has no zero in } (a, b)\} = (2/\pi) \arcsin \sqrt{a/b}$ . It is possible to also give an exact expression of the probability  $\mathbf{P}\{\inf_{a \leq t \leq b} |W(t) - M| = 0\}$ , although for  $M \neq 0$  this one is relatively more complicated. This is indicated in the Lemma below.

**Lemma 5** *Let  $0 < a \leq b < \infty$ . We have*

$$\begin{aligned} \mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| = 0\right\} &= -\sqrt{\frac{2}{\pi}} \frac{M}{\sqrt{a}} \int_{\sqrt{\frac{b-a}{b}}}^1 u e^{-\frac{(Mu)^2}{2a}} \left[ \int_{|x| \leq M\sqrt{\frac{1-u^2}{a}}} \frac{e^{-\frac{x^2}{2}} dx}{\sqrt{2\pi}} \right] du \\ &\quad + \left[ 1 - \left(\frac{2}{\pi}\right) e^{-\frac{M^2}{2a}} \arctan \sqrt{\frac{a}{b-a}} \right]. \end{aligned}$$

*In particular  $\mathbf{P}\{\inf_{a \leq t \leq b} |W(t)| = 0\} = 1 - \frac{2}{\pi} \arctan \sqrt{a/(b-a)}$ . And for every positive real  $c$*

$$\begin{aligned} \mathbf{P}\{0 < \inf_{a \leq t \leq b} |W(t)| < c\} &= 2 \int_0^{c/\sqrt{a}} \left(1 - 2\Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right)\right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\ &\quad + 4 \int_{c/\sqrt{a}}^\infty \left(\Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right) - \Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right)\right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du. \end{aligned}$$

*Proof.* Let  $M_1 = M/\sqrt{a}$ , and put  $F(s) = \int_{\mathbf{R}} 2\Psi(|w|s) e^{-\frac{(M_1+w)^2}{2}} \frac{dw}{\sqrt{2\pi}}$ . We have

$$\mathbf{P}\left\{\inf_{a \leq t \leq b} |W(t) - M| = 0\right\} = \int_{\mathbf{R}} 2\Psi\left(\frac{|v|}{\sqrt{b-a}}\right) \frac{e^{-\frac{(M+v)^2}{2a}}}{\sqrt{2\pi a}} dv$$

$$\begin{aligned}
&= \int_{\mathbf{R}} 2\Psi\left(|w|\sqrt{\frac{a}{b-a}}\right) \frac{e^{-\frac{(M_1+w)^2}{2}}}{\sqrt{2\pi}} dw \\
&= F\left(\sqrt{\frac{a}{b-a}}\right).
\end{aligned} \tag{9}$$

As  $\frac{\partial}{\partial s}\Psi(|w|s) = |w|\Psi'(|w|s) = -\frac{|w|}{\sqrt{2\pi}}e^{-(|w|s)^2/2}$ , we have

$$\begin{aligned}
\frac{\partial}{\partial s}F(s) &= \int_{\mathbf{R}} 2\frac{\partial}{\partial s}\Psi\left(|w|s\right) \frac{e^{-\frac{(M_1+w)^2}{2}}}{\sqrt{2\pi}} dw \\
&= -\sqrt{\frac{2}{\pi}} \int_{\mathbf{R}} |w| e^{-\frac{1}{2}[(ws)^2 + (M_1+w)^2]} \frac{dw}{\sqrt{2\pi}}.
\end{aligned}$$

But  $[(ws)^2 + (M_1 + w)^2] = [w\sqrt{s^2 + 1} + \frac{M_1}{\sqrt{s^2 + 1}}]^2 + \frac{M_1^2 s^2}{s^2 + 1}$ , hence

$$\begin{aligned}
\frac{\partial}{\partial s}F(s) &= -\sqrt{\frac{2}{\pi}} e^{-\frac{M_1^2 s^2}{2(s^2 + 1)}} \int_{\mathbf{R}} |w| e^{-\frac{1}{2}[w\sqrt{s^2 + 1} + \frac{M_1}{\sqrt{s^2 + 1}}]^2} \frac{dw}{\sqrt{2\pi}} \\
&= -\sqrt{\frac{2}{\pi}} \frac{e^{-\frac{M_1^2 s^2}{2(s^2 + 1)}}}{s^2 + 1} \int_{\mathbf{R}} |z| e^{-\frac{1}{2}[z + \frac{M_1}{\sqrt{s^2 + 1}}]^2} \frac{dz}{\sqrt{2\pi}} \\
&= -\sqrt{\frac{2}{\pi}} \frac{e^{-\frac{M_1^2 s^2}{2(s^2 + 1)}}}{s^2 + 1} \mathbf{E} \left| g - \frac{M_1}{\sqrt{s^2 + 1}} \right|,
\end{aligned} \tag{10}$$

where  $g$  denotes a Gaussian standard random variable. But for any real  $a$

$$\mathbf{E} |g + a| = |a| \int_{-|a|}^{|a|} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} + \sqrt{\frac{2}{\pi}} e^{-a^2/2}. \tag{11}$$

Hence

$$\begin{aligned}
\frac{\partial}{\partial s}F(s) &= -\sqrt{\frac{2}{\pi}} \frac{e^{-\frac{M_1^2 s^2}{2(s^2 + 1)}}}{s^2 + 1} \left\{ \frac{M_1}{\sqrt{s^2 + 1}} \int_{|x| \leq \frac{M_1}{\sqrt{s^2 + 1}}} \frac{e^{-x^2/2} dx}{\sqrt{2\pi}} + \sqrt{\frac{2}{\pi}} e^{-\frac{M_1^2}{2(s^2 + 1)}} \right\} \\
&= -\sqrt{\frac{2}{\pi}} \frac{M_1 e^{-\frac{M_1^2 s^2}{2(s^2 + 1)}}}{(s^2 + 1)^{3/2}} \int_{|x| \leq \frac{M_1}{\sqrt{s^2 + 1}}} \frac{e^{-x^2/2} dx}{\sqrt{2\pi}} - \frac{2}{\pi} \frac{e^{-M_1^2/2}}{s^2 + 1}.
\end{aligned} \tag{12}$$

If  $M = 0$ , this takes a much simplified form

$$\frac{\partial}{\partial s}F(s) = -\frac{2}{\pi} \frac{1}{s^2 + 1}.$$

Further  $F(0) = \int_{\mathbf{R}} e^{-\frac{(M_1+w)^2}{2}} \frac{dw}{\sqrt{2\pi}} = 1$ . Therefore

$$\begin{aligned}
F(s) - 1 &= -\sqrt{\frac{2}{\pi}} M_1 \int_0^s e^{-\frac{M_1^2 t^2}{2(t^2 + 1)}} \left[ \int_{|x| \leq \frac{M_1}{\sqrt{t^2 + 1}}} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \right] \frac{dt}{(t^2 + 1)^{3/2}} \\
&\quad - \frac{2}{\pi} e^{-M_1^2/2} \int_0^s \frac{dt}{t^2 + 1}
\end{aligned}$$

$$\begin{aligned}
&= -e^{-M_1^2/2} \left\{ \sqrt{\frac{2}{\pi}} M_1 \int_0^s \left[ \int_{|x| \leq \frac{M_1}{\sqrt{t^2+1}}} e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \right] \frac{e^{\frac{M_1^2}{2(t^2+1)}} dt}{(t^2+1)^{3/2}} \right. \\
&\quad \left. - \frac{2}{\pi} \arctan s \right\}. \tag{13}
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\mathbf{P}\left\{ \inf_{a \leq t \leq b} |W(t) - M| = 0 \right\} = F\left(\sqrt{\frac{a}{b-a}}\right) \\
&= -\sqrt{\frac{2}{\pi}} \frac{M}{\sqrt{a}} \int_{\sqrt{\frac{b-a}{b}}}^1 u e^{-\frac{(Mu)^2}{2a}} \left[ \int_{|x| \leq M\sqrt{\frac{1-u^2}{a}}} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \right] du \\
&\quad + \left[ 1 - \frac{2}{\pi} e^{-\frac{M^2}{2a}} \arctan \sqrt{\frac{a}{b-a}} \right]. \tag{14}
\end{aligned}$$

If  $M = 0$ , this is simplified into

$$\mathbf{P}\left\{ \inf_{a \leq t \leq b} |W(t)| = 0 \right\} = \left[ 1 - \frac{2}{\pi} \arctan \sqrt{\frac{a}{b-a}} \right]. \tag{15}$$

As concerning the second formula, it suffices to write

$$\begin{aligned}
\mathbf{P}\{0 < \inf_{a \leq t \leq b} |W(t)| \leq c\} &= \mathbf{P}\{ \inf_{a \leq t \leq b} |W(t)| \leq c \} - \mathbf{P}\{ \inf_{a \leq t \leq b} |W(t)| = 0 \} \\
&= \left( 1 - 2 \int_{\frac{c}{\sqrt{a}}}^{\infty} \left( 1 - 2\Psi\left(\frac{u\sqrt{a}-c}{\sqrt{b-a}}\right) \right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \right) \\
&\quad - \left( 1 - 2 \int_0^{\infty} \left( 1 - 2\Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right) \right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \right) \\
&= -2 \int_{\frac{c}{\sqrt{a}}}^{\infty} \left( 1 - 2\Psi\left(\frac{u\sqrt{a}-c}{\sqrt{b-a}}\right) \right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\
&\quad + 2 \int_0^{\infty} \left( 1 - 2\Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right) \right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\
&= 2 \int_0^{\frac{c}{\sqrt{a}}} \left( 1 - 2\Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right) \right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\
&\quad + 4 \int_{\frac{c}{\sqrt{a}}}^{\infty} \left[ \Psi\left(\frac{u\sqrt{a}-c}{\sqrt{b-a}}\right) - \Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right) \right] \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du.
\end{aligned}$$

■

Notice that  $\frac{\pi}{2} = \int_0^{\infty} \frac{dt}{(1+t^2)}$ , and so

$$\begin{aligned}
1 - \frac{2}{\pi} \arctan s &= \frac{2}{\pi} \left[ \int_0^{\infty} \frac{dt}{(1+t^2)} - \int_0^s \frac{dt}{(1+t^2)} \right] \\
&= \frac{2}{\pi} \int_s^{\infty} \frac{dt}{(1+t^2)} \leq C \min(1, s^{-1}).
\end{aligned}$$

We deduce the bound obtained in Lemma 2.2

$$\mathbf{P}\left\{ \inf_{a \leq t \leq b} |W(t)| = 0 \right\} \leq C \min\left(1, \sqrt{\frac{b-a}{a}}\right).$$

In what follows we shall be interested in finding estimates of the delicate random variable  $\beta_{[a,b]}^{-\alpha} \cdot \chi_{\{\beta_{[a,b]} > 0\}}$ ,  $0 \leq \alpha < 1$ , where we set

$$\beta_{[a,b]} := \inf_{a \leq t \leq b} |W(t)|.$$

**Proposition 6** *Let  $b > a > 0$ , and  $\eta > 0$ . Then,*

$$\mathbf{P}\{0 < \beta_{[a,b]} \leq \eta\} \leq \frac{16\eta}{\sqrt{2\pi}} \min\left(\frac{1}{\sqrt{b-a}}, \frac{1}{\sqrt{a}}\right).$$

Further, for any real  $\alpha$ ,  $0 \leq \alpha < 1$

$$\mathbf{E} \left\{ \frac{1}{\beta_{[a,b]}^\alpha} \cdot \chi_{\{\beta_{[a,b]} > 0\}} \right\} \leq \frac{47}{1-\alpha} \min\left(\frac{1}{\sqrt{b-a}}, \frac{1}{\sqrt{a}}\right) + 1.$$

The result gives a control which is uniform in  $b-a$ . We have  $\beta_{[a,b]} \rightarrow 0$  as  $b-a \rightarrow \infty$ , but in the same time the constraint  $\beta_{[a,b]} > 0$  becomes also stronger, making the probability of the set  $\{\beta_{[a,b]} > 0\}$  small. When  $b-a \rightarrow 0$ ,  $\beta_{[a,b]} \rightarrow |W(a)|$ , and this is reflected by the term  $1/\sqrt{a}$  in our estimate.

*Proof.* Write  $\beta = \beta_{[a,b]}$ . By using the second formula in Lemma 5

$$\begin{aligned} \mathbf{P}\{0 < \beta \leq \eta\} &= 4 \int_0^{\eta/\sqrt{a}} \left(1 - 2\Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right)\right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\ &\quad + 4 \int_{\eta/\sqrt{a}}^\infty \left(\Psi\left(\frac{u\sqrt{a}-\eta}{\sqrt{b-a}}\right) - \Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right)\right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du. \end{aligned} \quad (16)$$

As  $1 - 2\Psi(x) = \int_{-x}^x e^{-u^2/2} \frac{du}{\sqrt{2\pi}} \leq \min((2/\pi)^{1/2}x, 1)$ ,  $x \geq 0$

$$\begin{aligned} &\int_0^{\eta/\sqrt{a}} \left(1 - 2\Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right)\right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\ &\leq \int_0^{\eta/\sqrt{a}} \min\left(\left(\frac{2}{\pi}\right)^{1/2} \frac{u\sqrt{a}}{\sqrt{b-a}}, 1\right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\ &\leq \min\left(\left(\frac{2}{\pi}\right)^{1/2} \frac{\sqrt{a}}{\sqrt{b-a}} \int_0^{\eta/\sqrt{a}} u \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du, \frac{\eta}{\sqrt{2\pi a}}\right) \\ &\leq \min\left(\frac{\eta \max_{u \geq 0} u e^{-\frac{u^2}{2}}}{\pi \sqrt{b-a}}, \frac{\eta}{\sqrt{2\pi a}}\right) = \eta \min\left(\frac{1}{\pi \sqrt{b-a}}, \frac{1}{\sqrt{2\pi a}}\right). \end{aligned} \quad (17)$$

Furthermore

$$\Psi\left(\frac{u\sqrt{a}-\eta}{\sqrt{b-a}}\right) - \Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right) = \int_{\frac{u\sqrt{a}-\eta}{\sqrt{b-a}}}^{\frac{u\sqrt{a}}{\sqrt{b-a}}} e^{-\frac{u^2}{2}} \frac{du}{\sqrt{2\pi}} \leq \frac{\eta}{\sqrt{2\pi(b-a)}}, \quad (18)$$

which implies

$$\begin{aligned} \int_{\eta/\sqrt{a}}^\infty \left(\Psi\left(\frac{u\sqrt{a}-\eta}{\sqrt{b-a}}\right) - \Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right)\right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du &\leq \frac{\eta}{\sqrt{2\pi(b-a)}} \int_{\eta/\sqrt{a}}^\infty \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\ &\leq \frac{\eta}{\sqrt{2\pi(b-a)}}. \end{aligned} \quad (19)$$

Besides, with the variable change  $u = v\sqrt{\frac{b-a}{a}}$ , letting  $v_\eta = \frac{\eta}{\sqrt{b-a}}$ .

$$\begin{aligned} \int_{\eta/\sqrt{a}}^{\infty} \left( \Psi\left(\frac{u\sqrt{a}-\eta}{\sqrt{b-a}}\right) - \Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right) \right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\ = \sqrt{\frac{b-a}{a}} \int_{v_\eta}^{\infty} (\Psi(v-v_\eta) - \Psi(v)) \frac{e^{-\frac{v^2(b-a)}{2a}}}{\sqrt{2\pi}} dv. \end{aligned}$$

We have

$$\sqrt{\frac{b-a}{a}} \int_{v_\eta}^{2v_\eta} (\Psi(v-v_\eta) - \Psi(v)) \frac{e^{-\frac{v^2(b-a)}{2a}}}{\sqrt{2\pi}} dv \leq \sqrt{\frac{b-a}{a}} \frac{v_\eta}{\sqrt{2\pi}} = \frac{\eta}{\sqrt{2\pi a}}.$$

Now if  $v \geq 2v_\eta$ , then  $v - v_\eta \geq v - v/2 = v/2$ . Consequently

$$\begin{aligned} & \sqrt{\frac{b-a}{a}} \int_{2v_\eta}^{\infty} (\Psi(v-v_\eta) - \Psi(v)) \frac{e^{-\frac{v^2(b-a)}{2a}}}{\sqrt{2\pi}} dv \\ &= \sqrt{\frac{b-a}{a}} \int_{2v_\eta}^{\infty} \left\{ \int_{v-v_\eta}^v \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx \right\} \frac{e^{-\frac{v^2(b-a)}{2a}}}{\sqrt{2\pi}} dv \\ &\leq \sqrt{\frac{b-a}{a}} v_\eta \int_{2v_\eta}^{\infty} \frac{e^{-v^2/8}}{\sqrt{2\pi}} \frac{e^{-\frac{v^2(b-a)}{2a}}}{\sqrt{2\pi}} dv \\ &\leq \sqrt{\frac{b-a}{a}} v_\eta \int_0^{\infty} e^{-v^2/8} \frac{dv}{2\pi} = \frac{\eta}{\sqrt{2\pi a}}. \end{aligned}$$

It follows that

$$\begin{aligned} \int_{\eta/\sqrt{a}}^{\infty} \left( \Psi\left(\frac{u\sqrt{a}-\eta}{\sqrt{b-a}}\right) - \Psi\left(\frac{u\sqrt{a}}{\sqrt{b-a}}\right) \right) \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du &\leq \min\left(\frac{2\eta}{\sqrt{2\pi a}}, \frac{\eta}{\sqrt{2\pi(b-a)}}\right) \\ &\leq \frac{2\eta}{\sqrt{2\pi}} \min\left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b-a}}\right). \end{aligned} \quad (20)$$

By reporting

$$\begin{aligned} \mathbf{P}\{0 < \beta \leq \eta\} &\leq 4\eta \min\left(\frac{1}{\pi\sqrt{b-a}}, \frac{1}{\sqrt{2\pi a}}\right) + \frac{8\eta}{\sqrt{2\pi}} \min\left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b-a}}\right) \\ &\leq \frac{16\eta}{\sqrt{2\pi}} \min\left(\frac{1}{\sqrt{b-a}}, \frac{1}{\sqrt{a}}\right). \end{aligned} \quad (21)$$

Now let  $X$  be a random variable such that  $X \geq 0$  a.s. As  $\chi\{[0, 1]\} = \sum_{n=0}^{\infty} \chi\{[\frac{1}{2^{n+1}}, \frac{1}{2^n}]\}$ , we have by the Beppo-Levi Theorem

$$\mathbf{E} \frac{1}{X^\alpha} \cdot \chi\{X > 0\} = \sum_{n=0}^{\infty} \mathbf{E} \left( \frac{1}{X^\alpha} \cdot \chi\left\{\frac{1}{2^{n+1}} < X \leq \frac{1}{2^n}\right\} \right) + \mathbf{E} \frac{1}{X^\alpha} \cdot \chi\{X > 1\}, \quad (22)$$

where "=" means that both expressions are simultaneously finite or infinite. And

$$\begin{aligned} & \sum_{n=0}^{\infty} \mathbf{E} \left( \frac{1}{X^\alpha} \cdot \chi\left\{\frac{1}{2^{n+1}} < X \leq \frac{1}{2^n}\right\} \right) + \mathbf{E} \frac{1}{X^\alpha} \cdot \chi\{X > 1\} \\ &\leq \sum_{n=0}^{\infty} 2^{\alpha(n+1)} \mathbf{P}\{0 < X \leq \frac{1}{2^n}\} + 1. \end{aligned} \quad (23)$$



Apply this with  $X = \beta$ . Then  $\mathbf{E} \beta^{-\alpha} \cdot \chi\{\beta > 0\}$  will be finite once we prove that the series

$$\sum_{n=0}^{\infty} 2^{\alpha(n+1)} \mathbf{P}\{0 < \beta \leq \frac{1}{2^n}\}$$

is convergent. By using (2.13) with  $\eta = 2^{-n}$ , we get

$$\begin{aligned} \sum_{n=0}^{\infty} 2^{\alpha(n+1)} \mathbf{P}\{0 < \beta \leq \frac{1}{2^n}\} &\leq \frac{2^{\alpha+4}}{\sqrt{2\pi}} \min\left(\frac{1}{\sqrt{b-a}}, \frac{1}{\sqrt{a}}\right) \sum_{n=0}^{\infty} 2^{-n(1-\alpha)} \\ &\leq \frac{32}{2^{1-\alpha} - 1} \min\left(\frac{1}{\sqrt{b-a}}, \frac{1}{\sqrt{a}}\right) \\ &\leq \frac{47}{(1-\alpha)} \min\left(\frac{1}{\sqrt{b-a}}, \frac{1}{\sqrt{a}}\right). \end{aligned} \quad (24)$$

And we conclude that

$$\mathbf{E} \frac{1}{\beta^\alpha} \cdot \chi\{\beta > 0\} \leq \frac{47}{1-\alpha} \min\left(\frac{1}{\sqrt{b-a}}, \frac{1}{\sqrt{a}}\right) + 1,$$

as claimed. ■

### 3 Proof of Theorem

Recall that  $I_N = [T_N, T_{N+1}]$ . It is now easy. As

$$\sum_{N \geq 1} \mathbf{P}\{0 < \beta_{I_N} \leq \eta_N\} \leq C \sum_{N \geq 1} \eta_N \min\left(\frac{1}{\sqrt{T_{N+1} - T_N}}, \frac{1}{\sqrt{T_N}}\right) < \infty,$$

we deduce from Borel-Cantelli Lemma that

$$\mathbf{P}\left\{\inf_{t \in I_N} |W(t)| \geq \eta_N \text{ or } W(t) = 0 \text{ for some } t \in I_N, \quad N \text{ ultimately}\right\} = 1.$$

■

*Acknowledgments.* I thank Istvan Berkes for valuable remarks and careful reading of the paper.

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